A typing system for the modalized Heyting calculus (extended abstract)

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I discuss a typing system for the modalized Heyting calculus **mHC** introduced by Leo Esakia (following on earlier work by Kuznetsov and Muravitsky). Its specific axiom(s) can be described as "modally guarded" classical tautologies. As it turns out, the most suitable proof term assignment for this system appears to be a fine-grained (or guarded) version of the static catch/throw system of Crolard.

1 Preliminaries on mHC

1.1 Syntax

(Intuitionistic) modal formulas over a supply of propositional variables Σ are defined by

$$A, B ::= \bot \quad | \quad p \quad | \quad A \to B \quad | \quad A \wedge B \quad | \quad A \vee B \quad | \quad \rhd A$$

where $p \in \Sigma$. The set is denoted by $\mathcal{L}_{\rhd\mathsf{int}}\Sigma$, but unless explicitly stated otherwise, I will keep Σ fixed throughout and drop it from the notation. Define $\mathsf{int} := \{\bot, \to, \wedge, \vee\}$ and $\rhd\mathsf{int} := \mathsf{int} \cup \{\rhd\}$. Given any $C \subseteq \rhd\mathsf{int}$, $\mathcal{L}_C$ will denote the fragment of $\mathcal{L}_{\rhd\mathsf{int}}$ obtained using only the connectives in C.

The Hilbert-style deduction system $\vdash_{\mathbf{hm}}$ for **mHC** given in Table 1. As usual, $\mathbf{K}^{\mathsf{i}}$ is the system which from axioms for $\rhd$ in Table 1 uses only (**nrm**)—i.e., no (**r**), no (**next**) and no (**derv**). Following the standard notation of Chagrov and Zakharyaschev [7], given a logic $\Gamma \subseteq \mathcal{L}_C$ and $\alpha \in \mathcal{L}_C$, we denote by $\Gamma \oplus \alpha$ the system obtained by closing the set $\Gamma \cup \{\alpha\}$ under **MP** and **NEC**.

Remark 1. *Equivalence of axioms from Table 2 can of course be proved without assuming any other axioms and rules governing modality in 1—not even normality. See also Remark 9.*

Remark 2. *Note that the axioms in Table 2 share a common form: apart from having identical antecedent, their consequent is an axiom for classical logic, with B playing the role of* $\bot$*. However, not all the axioms of classical logic would be equally suitable as these consequents. The fine-grained analysis of [1, 2] is particularly useful here: we cannot use any axiom which—when B is read as* $\bot$*—would imply* `ex falso quodlibet`*. This will be discussed in the full paper.*

1.2 Semantics of mHC

There is no space to discuss the details here, especially as all the semantics facts (I mean the semantics of provable propositions, not the semantics of proofs) we need here have been established a long time ago. The reader is referred to [10, 11, 17] for overview. Briefly speaking, in a Kripke frame for the intuitionistic propositional logic, i.e., a poset order, we interpret a modality of **mHC** type by the strict part of the poset order (plus possibly some reflexive loops) and in topological spaces a canonical interpretation is provided by the so-called Cantor-Bendixson derivative. In fact, **mHC** seems to have been born out of semantic rather than syntactic considerations.

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Table 1: The Hilbert-style deduction $\vdash_{\mathbf{hm}}$ for **mHC**

Axioms of the intuitionistic propositional calculus, see, e.g., [7, Sec. 1.3,(A1)-(A9)]

Axioms for $\rhd$

$$(\mathbf{nrm}) \quad \rhd(A \to B) \to (\rhd A \to \rhd B)$$

$$(\mathbf{r}) \quad A \to \rhd A$$

An arbitrarily chosen axiom from Table 2

Inference rule for int	Inference rule for $\rhd$
MP $\dfrac{A \to B, \quad A}{B}$	**NEC** $\dfrac{A}{\rhd A}$

Table 2: Equivalent **mHC**-specific axioms

$\rhd$(**peirce**)	$\rhd B \to (((A \to B) \to A) \to A)$
$\rhd$(**em**)	$\rhd B \to A \vee (A \to B)$
$\rhd$(**dMn&n**)	$\rhd B \to ((A_1 \to B) \to (A_2 \to B) \to B) \to A_1 \vee A_2 \vee B$
$\rhd$($\to$ **to**$\vee$)	$\rhd B \to (A_1 \to A_2) \to (A_1 \to B) \vee A_2$

1.3 Canonical modality

In any system extending the intuitionistic logic with propositional quantifiers—in particular, in the type system of Girard-Reynolds' system F—a "canonical" **mHC** modality is available. As far as I know, this observation is essentially due to Leo Esakia and collaborators, see [10, 11]. The definition is

$$\blacktriangleright B := \forall A.((A \to B) \vee A).$$

Proposition 3. $\blacktriangleright$ *satisfies all the axioms of* **mHC**.

Proof. An exercise, see also Remark 9. □

Proposition 4. *For every $\rhd$ satisfying any of the equivalent axioms in Table 2, $\rhd A \to \blacktriangleright A$ is intuitionistically derivable*

Proof. An exercise, see also Remark 9. □

Note this last observation does not require any other $\rhd$ axioms or rules from Table 1. It holds simply by properties of universal quantification. Also note that in systems with infinitary meets, e.g., in any locale, $\bigwedge$ can replace $\forall$ in the definition of $\blacktriangleright$.

The material so far has been known, even if perhaps not too widely. Now it's time for our contributions.

Table 3: Natural deduction system $\vdash_{\mathbf{dm}}$ for **mHC**

$$\Gamma, x : A \vdash_{\mathbf{dm}} x : A \ .\Delta \qquad \text{(start)}$$

$$\frac{\Gamma \vdash_{\mathbf{dm}} M : A \ .\Delta \qquad \Gamma \vdash_{\mathbf{dm}} N : \rhd B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \langle M \rangle \uparrow \{N\} \alpha : B \ . \succ\!\!\rhd(B).\alpha : A\,\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : A \ . \succ\!\!\rhd(B).\alpha : A\,\Delta \qquad \Gamma \vdash_{\mathbf{dm}} N : \rhd B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \alpha \downarrow \{N\}[M] : A \ .\Delta} \qquad (\mathsf{R}\rhd)$$

$$\frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash_{\mathbf{dm}} N : B \ .\Delta \qquad \Gamma \vdash_{\mathbf{dm}} M_1 : \rhd A_1 \ .\Delta \quad \ldots \quad \Gamma \vdash_{\mathbf{dm}} M_n : \rhd A_n \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \mathsf{box}\, N[\rhd(x_1, \ldots, x_n) \rightsquigarrow (M_1, \ldots, M_n)] : \rhd B \ .\Delta} \qquad (\mathsf{R}\rhd)$$

$$\frac{\Gamma, x : A \vdash_{\mathbf{dm}} M : B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \lambda x.M : A \to B \ .\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : A \to B \ .\Delta \qquad \Gamma \vdash_{\mathbf{dm}} N : A \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} MN : B \ .\Delta} \qquad (\mathsf{R}\to)$$

$$\frac{\Gamma \vdash_{\mathbf{dm}} L : A \vee B \ .\Delta \qquad \Gamma, x : A \vdash_{\mathbf{dm}} M : C \ .\Delta \qquad \Gamma, y : B \vdash_{\mathbf{dm}} N : C \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \mathsf{case}\, L\, \mathsf{of}\, [x \rightsquigarrow M \mid y \rightsquigarrow N] : C \ .\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : A \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \sqcup_1 M : A \vee B \ .\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \sqcup_2 M : A \vee B \ .\Delta} \qquad (\mathsf{R}\vee)$$

$$\frac{\Gamma \vdash_{\mathbf{dm}} M : A \ .\Delta \qquad \Gamma \vdash_{\mathbf{dm}} N : B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \langle M, N \rangle : A \wedge B \ .\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : A \wedge B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \sqcap_1 M : A \ .\Delta} \qquad \frac{\Gamma \vdash_{\mathbf{dm}} M : A \wedge B \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \sqcap_2 M : B \ .\Delta} \qquad (\mathsf{R}\wedge)$$

$$\frac{\Gamma \vdash_{\mathbf{dm}} M : \bot \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \mathsf{abort}\, M : B \ .\Delta} \qquad (\mathsf{R}\bot)$$

2 Natural Deduction

Through the section, we fix a supply of ordinary (proof-term) variables $x, y, z \ldots$ and *stack* or *continuation* variables $\alpha, \beta, \gamma \ldots$. Proof terms of the natural deduction system are defined by the following grammar:

$$\begin{array}{rcll}
M, N, L ::= & x & \mid \ \langle M \rangle \uparrow \{N\} \alpha \quad \mid \quad \alpha \downarrow \{N\}[M] & (\mathsf{T}\rhd) \\
& & \mid \ \mathsf{box}\, N[\rhd(x_1, \ldots, x_n) \rightsquigarrow (M_1, \ldots, M_n)] & (\mathsf{T}\rhd) \\
& & \mid \ \lambda x.M \quad \mid \quad MN & (\mathsf{T}\to) \\
& & \mid \ \mathsf{case}\, L\, \mathsf{of}\, [x \rightsquigarrow M \mid y \rightsquigarrow N] \quad \mid \quad \sqcup_1 M \quad \mid \quad \sqcup_2 N & (\mathsf{T}\vee) \\
& & \mid \ \langle M, N \rangle \quad \mid \quad \sqcap_1 M \quad \mid \quad \sqcap_2 N & (\mathsf{T}\wedge) \\
& & \mid \ \mathsf{abort}\, M & (\mathsf{T}\bot)
\end{array}$$

For any $C \subseteq \rhd\mathsf{int}$, the proof terms typed by the relation $\vdash_{\mathbf{dm}C}$ (i.e., the restriction of $\vdash_{\mathbf{dm}}$ to connectives in C) defined below are those included in $(\mathsf{T}*)$, where $* \in C$, plus those in $(\mathsf{T}\alpha)$.

Define the notions of

$$\begin{array}{llcll}
\text{context} & \Gamma & ::= & \varnothing & \mid \quad \Gamma, x : A \\
\text{mg-cocontext} & \Delta & ::= & \varnothing & \mid \quad \Delta \succ\!\!\rhd(B).\gamma : C
\end{array}$$

In the above, "mg" stands for "modally guarded". Note that mg-cocontexts cannot be reordered: $\succ\!\!\rhd$ is not commutative (or associative) in any meaningful sense! The general form of a (pseudo-)judgment is $\Gamma \vdash_{\mathbf{dm}} N : A \ .\Delta$. The actual judgements are given in Table 3.

2.1 Intuitive reading and comments on the proof system

$\langle M \rangle \uparrow \{N\} \alpha$ should be read as *throw M to α, with N witnessing*. $\alpha \downarrow \{N\}[M]$ should be read as *catch α by M, with N witnessing*. Clearly, the witnessing part seems to be the novel one. Note that catch and throw are used here in the sense of Crolard [8], i.e., these are more like static operators a la Scheme rather than dynamic exceptions. We see them in action below; a few words on the modal aspects of our system first.

The rule for $\rhd$ is a variant of the standard one proposed first by Bellin [4] for single-context deduction systems for modal logic, see [13, 20]. The only difference is that we do not have to discharge Γ in any of the premises because of the presence of $(\mathbf{r})$ in our axiomatization. This means that we can define $\mathsf{delay}\, M := \mathsf{box}\, M[\rhd() \rightsquigarrow ()]$ and derive

$$\frac{\Gamma \vdash_{\mathbf{dm}} M : A \ .\Delta}{\Gamma \vdash_{\mathbf{dm}} \mathsf{delay}\, M : \rhd A \ .\Delta}$$

In the original setup of [13, 20], i.e., in proof systems for $\mathbf{K}^{\mathsf{i}}$, this is only derivable for these $M : A$ which are typable already in the empty context. The derivability of $\mathsf{delay}\, M$ and normality law for the modality turns the continuation-free part of this proof system (i.e., with mg-cocontexts and throwing/catching rules removed) into a generic deduction system for *applicative functors* of McBride and Paterson [19].

Note also that if the present form of $\mathsf{box}\, N[\rhd(x_1, \ldots, x_n) \rightsquigarrow (M_1, \ldots, M_n)]$ with *explicit substitutions* (see the discussion in [9]) is not deemed legible or elegant enough, we can also formulate our deduction system in the *Fitch-* or *multi-context style* a la [6, 9, 18], popular also in more recent references such as [15, 16]. Details will be provided in the full paper.

2.2 Examples of typable programs

The details of all derivations mentioned in this section can be found in the appendix. See also Remark 9.

Example 5. *First, let us see how proofs of some of equivalent axioms from Table 2 above can look like.*

$$\mathsf{proof}\rhd(\mathbf{peirce}) := \lambda x.\lambda y.\alpha\!\downarrow\!\{x\}[(y(\lambda z.\langle z\rangle\!\uparrow\!\{x\}\alpha)]$$
$$\mathsf{proof}\rhd(\mathbf{em}) := \lambda x.\alpha\!\downarrow\!\{x\}[\ \amalg_2\, \lambda y.\langle \amalg_1\, y\rangle\!\uparrow\!\{x\}\alpha]$$

The reader is invited to work out proofs for remaining formulas from Table 2 as an entertaining and enlightening exercise.

Example 6. *To see an example of a slightly different and rather non-trivial nature, assume we have a combinator for an axiom modal logicians call* $(.\mathbf{3})$ *or simply* $(\mathbf{lin})$ *for linearity (of the accessibility relation interpreting the modal operator):*

$$\mathsf{c}(\mathbf{lin}) : \rhd(A \to B) \vee \rhd(B \to A).$$

Given such a combinator, we can derive a proof for the Gödel-Dummet law $(\mathbf{gd})$ *(see [3, 12] for its Curry-Howard interpretation and computational importance).*

$$\begin{aligned}\mathsf{der\_comb}(\mathbf{gd}) := \mathsf{case}\ \mathsf{c}(\mathbf{lin})\ \mathsf{of}\ [\\ &x \rightsquigarrow \alpha\!\downarrow\!\{x\}[\ \amalg_1\, \lambda y.((\langle \amalg_2\, \lambda z.y\rangle\!\uparrow\!\{x\}\alpha)y)]\\ &\mid x \rightsquigarrow \alpha\!\downarrow\!\{x\}[\ \amalg_2\, \lambda y.((\langle \amalg_1\, \lambda z.y\rangle\!\uparrow\!\{x\}\alpha)y)]]\end{aligned}$$

The interesting feature of this derivation is that assuming a modal principle, we obtain a purely propositional law, i.e., in the language $\mathcal{L}_{\mathsf{int}}$. This shows again the somewhat non-standard character of the **mHC** *modality and its close connection with the intuitionistic poset order/the structure of Heyting algebra: "ordinary" modal axioms tend to combine without "side effects" for $\mathcal{L}_{\mathsf{int}}$. An additional reason for interest in linearity axioms is that they tend to be valid in models considered in recent Theoretical Computer Science, for example the one of [5].*

Example 7. *As our final example assume we have a combinator for the axiom* $(\mathbf{ver})$*, which is just* $\rhd A$*. This axioms forces that* $\rhd$ *is a verum operator: if it is assumed, any formula prefixed with* $\rhd$ *becomes a tautology. In fact, one can either assume* $\mathsf{c}(\mathbf{ver}) : \rhd A$ *for each* A *or equivalently assume just* $\mathsf{c}(\mathbf{ver}_\bot) : \rhd\bot$ *and then obtain* $(\mathbf{ver})$ *as follows:*

$$\frac{\mathsf{c}(\mathbf{ver}_\bot) : \rhd\bot \vdash_{\mathbf{dm}} \mathsf{c}(\mathbf{ver}_\bot) : \rhd\bot \ .\varnothing \qquad x : \bot \vdash_{\mathbf{dm}} \mathsf{abort}\, x : A \ .\varnothing}{\mathsf{c}(\mathbf{ver}_\bot) : \rhd\bot \vdash_{\mathbf{dm}} \mathsf{box}\ \mathsf{abort}\, x[\rhd(x) \rightsquigarrow (\mathsf{c}(\mathbf{ver}_\bot))] : \rhd A \ .\varnothing}$$

Note it is the first time we see a "typical" modal rule in action. Note also that this derivation relies crucially on the law `ex falso quodlibet`*, i.e., the* $(\mathsf{R}\bot)$ *row in our Table 3. Hence, these two combinators will not be equivalent over Johansson's minimal logic discussed in [1, 2].*

Now with the benefit of Example 5, it is almost trivial to derive a combinator for an arbitrarily chosen classical axiom from $\mathsf{c}(\mathbf{ver})$*. Here is for example a proof term for the Peirce axiom:*

$$\lambda y.\alpha{\downarrow}\{\mathsf{c}(\mathbf{ver})\}[(y(\lambda z.\langle z\rangle{\uparrow}\{\mathsf{c}(\mathbf{ver})\}\alpha)]$$

As there are no restriction of witnessing use of $\mathsf{c}(\mathbf{ver})$*, its addition would make our typing system essentially collapse to that of [8], with no restrictions on throwing and catching rules.*

2.3 Soundness Theorem

To claim that $\vdash_{\mathbf{dm}}$ is an adequate deduction system for **mHC**, we need to know if it coincides with $\vdash_{\mathbf{hm}}$ as far as derivable judgements or inhabited types are concerned. For any $\Gamma := x_1 : A_1, \ldots, x_n : A_n$, define $\bigwedge\Gamma := A_1 \wedge \cdots \wedge A_n$. Then we have

Theorem 8. *For any* $A \in \mathcal{L}_{\rhd\mathsf{int}}$*, any proof term* M *any* Γ *and any*

$$\Delta := \rightarrowtail\!\rhd(B_1).\gamma_1 : C_1 \ \ldots \ \rightarrowtail\!\rhd(B_m).\gamma_m : C_m \ ,$$

we have that

$$\Gamma \vdash_{\mathbf{dm}} M : A \ .\Delta \quad \textit{iff} \quad \vdash_{\mathbf{hm}} \rhd B_m \to C_m \vee (\ldots(\rhd B_1 \to C_1 \vee (\bigwedge\Gamma \to A))\ldots)$$

Proof. As usual, by induction on complexity of derivations. Details will be provided in the full version. □

3 Ongoing and future work

3.1 Reduction and conversion rules

So far, I have not said anything about actual reduction and conversion rules. These will have to suitably combine these of [13] for modality and these of [8] for catch and throw operators. Furthermore, one should choose an appropriate notion of a normal form which would allow to state and prove a normalization theorem. This will be discussed at length in the final version of the paper. For the time being, just note that a path to prove a normalization result is already indicated in Section 1.3.

3.2 Encoding of Crolard's original calculus

This is something already hinted at in Example 7. Basically, we just need to ensure that after the addition of $\mathsf{c}(\mathbf{ver})$, the reduction and conversion rules are essentially the same as those of [8]. Details will be provided in the full paper.

3.3 Computational meaning

The most important question, however, is what is the real computation relevance of our calculus? In particular, is there any connection between our guarded continuations and *delimited* continuations?

At any rate, I hope that the present study indicates there is a lot of interesting research to be done on the intersection between modal logic and control operators. Apart from some notable and laudable exceptions like [13, 14], there do not seem to be too many researchers working in the area presently.

Remark 9. *Most of syntactic Hilbert-style derivations relevant for this paper have been formalized in a Coq file associated with [17]. While the work is still in progress (and I am a Coq novice), the file has been temporarily made available under hidden link*
www8.informatik.uni-erlangen.de/~litak/papers/esakia_provability_smack.v.

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Appendix

Derivation of proof $\rhd(\mathbf{peirce})$. Denote $\Gamma := x : \rhd B, y : (A \to B) \to A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, z : A \vdash_{\mathbf{dm}} z : A \;.\varnothing \qquad \Gamma, z : A \vdash_{\mathbf{dm}} x : \rhd B \;.\varnothing}{\Gamma, z : A \vdash_{\mathbf{dm}} \langle z \rangle \uparrow \{x\} \alpha : B \;.\rightarrowtail\!\rhd(B).\alpha : A}}{\Gamma \vdash_{\mathbf{dm}} \lambda z.\langle z \rangle \uparrow \{x\} \alpha : A \to B \;.\rightarrowtail\!\rhd(B).\alpha : A}}{\Gamma \vdash_{\mathbf{dm}} y(\lambda z.\langle z \rangle \uparrow \{x\} \alpha) : A \;.\rightarrowtail\!\rhd(B).\alpha : A} \qquad \Gamma \vdash_{\mathbf{dm}} x : \rhd B \;.\varnothing}{\dfrac{\Gamma \vdash_{\mathbf{dm}} \alpha \downarrow \{x\} [y(\lambda z.\langle z \rangle \uparrow \{x\} \alpha)] : A \;.\varnothing}{\varnothing \vdash_{\mathbf{dm}} \lambda x.\lambda y.\alpha \downarrow \{x\} [(y(\lambda z.\langle z \rangle \uparrow \{x\} \alpha))] : \rhd B \to (((A \to B) \to A) \to A) \;.\varnothing}}$$

Derivation of proof$\rhd$(**em**). Denote $\Gamma := x : \rhd B, y : A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\mathbf{dm}} y : A \;.\varnothing}{\Gamma \vdash_{\mathbf{dm}} \sqcup_1 y : A \vee (A \to B) \;.\varnothing} \qquad \Gamma \vdash_{\mathbf{dm}} x : \rhd B \;.\varnothing}{\Gamma \vdash_{\mathbf{dm}} \langle \sqcup_1 y \rangle \uparrow \{x\} \alpha : B \;.\rightarrowtail\!\rhd(B).\alpha : A \vee (A \to B)}}{x : \rhd B \vdash_{\mathbf{dm}} \lambda y.\langle \sqcup_1 y \rangle \uparrow \{x\} \alpha : A \to B \;.\rightarrowtail\!\rhd(B).\alpha : A \vee (A \to B)}}{x : \rhd B \vdash_{\mathbf{dm}} \sqcup_2 \lambda y.\langle \sqcup_1 y \rangle \uparrow \{x\} \alpha : A \vee A \to B \;.\rightarrowtail\!\rhd(B).\alpha : A \vee (A \to B)} \qquad x : \rhd B \vdash_{\mathbf{dm}} x : \rhd B \;.\varnothing}{\dfrac{x : \rhd B \vdash_{\mathbf{dm}} \alpha \downarrow \{x\} [\sqcup_2 \lambda y.\langle \sqcup_1 y \rangle \uparrow \{x\} \alpha] : A \vee (A \to B) \;.\varnothing}{\varnothing \vdash_{\mathbf{dm}} \lambda x.\alpha \downarrow \{x\} [\sqcup_2 \lambda y.\langle \sqcup_1 y \rangle \uparrow \{x\} \alpha] : \rhd B \to A \vee (A \to B) \;.\varnothing}}$$

Derivation of der_comb(**gd**) with c(**lin**) assumed. Denote $\Gamma := x : \rhd(A \to B), y : A$. We will write (**gd**) rather than $(A \to B) \vee (B \to A)$ for readability. Then the crucial part of derivation is (the rest being a standard of case split):

$$\dfrac{\dfrac{\dfrac{\Gamma, z : B \vdash_{\mathbf{dm}} y : A \;.\varnothing}{\Gamma \vdash_{\mathbf{dm}} \lambda z.y : B \to A \;.}}{\Gamma \vdash_{\mathbf{dm}} \sqcup_2 \lambda z.y : (\mathbf{gd}) \;.\varnothing} \qquad \Gamma \vdash_{\mathbf{dm}} x : \rhd(A \to B) \;.\varnothing}{\dfrac{\Gamma \vdash_{\mathbf{dm}} \langle \sqcup_2 \lambda z.y \rangle \uparrow \{x\} \alpha : A \to B \;.\rightarrowtail\!\rhd(A \to B).\alpha : (\mathbf{gd})}{\dfrac{\Gamma \vdash_{\mathbf{dm}} (\langle \sqcup_2 \lambda z.y \rangle \uparrow \{x\} \alpha) y : B \;.\rightarrowtail\!\rhd(A \to B).\alpha : (\mathbf{gd})}{\dfrac{x : \rhd(A \to B) \vdash_{\mathbf{dm}} \lambda y.((\langle \sqcup_2 \lambda z.y \rangle \uparrow \{x\} \alpha) y) : A \to B \;.\rightarrowtail\!\rhd(A \to B).\alpha : (\mathbf{gd})}{\dfrac{x : \rhd(A \to B) \vdash_{\mathbf{dm}} \sqcup_1 \lambda y.((\langle \sqcup_2 \lambda z.y \rangle \uparrow \{x\} \alpha) y) : (\mathbf{gd}) \;.\rightarrowtail\!\rhd(A \to B).\alpha : (\mathbf{gd})}{x : \rhd(A \to B) \vdash_{\mathbf{dm}} \alpha \downarrow \{x\} [\sqcup_1 \lambda y.((\langle \sqcup_2 \lambda z.y \rangle \uparrow \{x\} \alpha) y)] : (\mathbf{gd}) \;.\varnothing}}}}}$$