A Model of Control Operators in Coherence Spaces giving rise to a New Model of Classical Set Theory

Thomas Streicher

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We revisit domain-theoretic models D of λ_{cc}, i.e. *λ-calculus with control*, and study the ensuing *classical realizability* models in the sense of J.-L. Krivine.

For this purpose, however, we have to identify a submodel $P \subseteq D$ of *error-free* elements. The existence of such a submodel is guaranteed by a theorem of A. Pitts.

If we start from a model D for λ_{cc} in Scott domains we just get $\mathbf{Set}$, i.e. nothing new. However, when starting from a model in $\mathbf{Coh}$, i.e. coherence spaces and stably continuous maps, we get a new boolean topos providing a model of ZF with restricted choice.

Domain Models of λ_{cc} (1)

Consider $D \cong \Sigma^{D^\omega}$ in **some** category of domains. Since $D \cong D^D$ the domain D gives rise to a model of *untyped λ-calculus* and thus to a *partial combinatory algebra* (pca).

In [SR98] it was shown that D also allows one to interpret control operators as

$$\mathsf{cc}(t.\vec{s}) = t(\mathsf{k}_{\vec{s}}.\vec{s}) \qquad \text{where} \qquad \mathsf{k}_{\vec{s}}(t.\vec{r}) = t(\vec{s})$$

where $\vec{s}$ and $\vec{r}$ are elements of D^ω thought of as *continuations*.

The domain $\Sigma = \{\bot \sqsubset \top\}$ is the domain of *observations* where $\bot$ represents *nontermination* and $\top$ is thought of as an *error* element.

Domain Models of λ_{cc} (2)

When a "program" $t \in D$ meets a "continuation" $\vec{s} \in D^\omega$ it results in an "observation" $t(\vec{s}) \in \Sigma$.
The set $⫫ = \{(t, \vec{s}) \mid t(\vec{s}) = \top\}$ is called **pole** and gives rise to a *Galois connection* between sets of programs and sets of continuations

$$A^{⫫} = \{\vec{s} \in D^\omega \mid \forall t \in A.\ t \mathrel{⫫} \vec{s}\} \qquad C^{⫫} = \{t \in D \mid \forall \vec{s} \in C.\ t \mathrel{⫫} \vec{s}\}$$

where $A \subseteq D$ and $C \subseteq D^\omega$.

In Krivine's *Classical Realizability* propositions are *biorthogonally closed* subsets of D, i.e. $A \subseteq D$ with $A^{⫫⫫} = A$. Thus propositions are always inhabited by $\top_D = \lambda\vec{s}.\top \in D$. For this reason we need a

Submodel P of "error-free" programs

By a theorem of A. Pitts there exists a unique subset P of D with

$$t \in P \qquad \text{iff} \qquad \forall \vec{s} \in P^\omega .\, t(\vec{s}) = \bot$$

i.e. $t \in P$ iff from $t(\vec{s}) = \top$ it follows that some $s_n \notin P$.
The elements of P are called "error-free" or "proof-like".

One easily shows that P is a subpca of D.

If $C \subseteq D^\omega$ with $C \cap P^\omega \neq \emptyset$ then $C^{\perp\!\!\!\perp} \cap P = \emptyset$, i.e. the proposition $C^{\perp\!\!\!\perp}$ does not have a proof-like realizer, which prevents us from inconsistency.

But before diving into classical realizability we introduce the *relative realizability* model RT(D, P) whose logic is still *intuitionistic*.

The Relative Realizability Model (1)

Its propositions are subsets of D. For $\varphi, \psi \in \mathcal{P}(D)$ let

$$\varphi \to \psi = \{t \in D \mid \forall s \in \varphi.\, ts \in \psi\}$$

For predicates $\varphi, \psi \in \mathcal{P}(D)^I$ on set I we define entailment as

$$\varphi \vdash_I \psi \qquad \text{iff} \qquad \exists t \in P. \forall i \in I. \forall s \in \varphi_i.\, ts \in \psi_i$$

i.e. iff $\bigcap\limits_{i \in I} \varphi_i \to \psi_i$ has non-empty intersection with P.

For $\varphi \in \mathcal{P}(D)^{I \times J}$ we define universal quantification as

$$\forall i{:}I.\varphi(i,j) \quad = \quad \bigcap_{i \in I} \varphi(i,j)$$

For a set I the equality predicate $eq_I \in \mathcal{P}(D)^{I \times I}$ on I is defined as

$$eq_I(i,j) = \{d \in D \mid i = j\}$$

The Relative Realizability Model (2)

à la Russell-Prawitz one may define the remaining connectives by their second order encoding. This gives rise to a **tripos** $\mathscr{P}$ over $\mathbf{Set}$ whose fibre over I is $\mathcal{P}(D)^I$ pre-ordered by $\vdash_I$.

From $\mathscr{P}$ like from any tripos we can construct the associated topos which in our case we call $\mathcal{E} = \mathsf{RT}(D, P)$.
Objects of $\mathcal{E}$ are pairs $X = (|X|, E_X)$ where $|X|$ is the underlying set and $E_X \in \mathcal{P}(D)^{|X|\times|X|}$ is symmetric and transitive in the sense of $\mathscr{P}$.
A morphism from X to Y is given by a predicate $F \in \mathcal{P}(D)^{|X|\times|Y|}$ s.t.

$$F(x,y) \vdash E_X(x,x) \wedge E_Y(y,y) \qquad F(x,y) \wedge E_X(x,x') \wedge E_Y(y,y') \vdash F(x',y')$$
$$F(x,y) \wedge F(x,y') \vdash E_Y(y,y') \qquad E_X(x,x) \vdash \bigcup_{y\in|Y|} F(x,y)$$

holds in the sense of $\mathscr{P}$. Logically equivalent F's are identified.

$\mathcal{E}$ hosts a model of IZF

On the class V of all sets we define $\mathcal{P}(D)$-valued binary predicates $\subseteq$ and $\in$ by mutual transfinite recursion as follows

$$x \subseteq y \equiv \forall z \in V.\, x(z) \to z \in y \qquad x \in y \equiv \exists z \in V.\, x = z \wedge y(z)$$

where $x = y$ stands for $x \subseteq y \wedge y \subseteq x$ and $x(z) = \{t \in D \mid \langle z, t \rangle \in x\}$.

One can show that this model validates all axioms of IZF, i.e. **Intuitionistic Zermelo Fraenkel Set Theory**.

This model construction is reminiscent of Scott and Solovay's **boolean valued models** but order on predicates is *uniform* and not pointwise.

Getting Boolean (1)

In $\mathcal{E}$ we have $true = D$ and $false = \emptyset$.
But there is a further truth value $U = \{\top_D\}$.
We define $\neg_U : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$ as $\neg_U A = A \to U$.
Then $j_U = \neg_U \circ \neg_U$ is a *Lawvere-Tierney topology on* $\mathcal{E}$.
The topos $\mathcal{E}_U = \mathcal{E}_{j_U}$ of j_U-sheaves is a boolean.

For staying "self-contained" we give an "elementary" reconstruction. Consider the subtripos $\mathscr{P}_U$ of $\mathscr{P}$ consisting of those $\varphi \in \mathcal{P}(D)^I$ such that $\neg_U \neg_U \varphi \vdash_I \varphi$ holds in $\mathscr{P}$. They are closed under implication and universal quantification.
Notice that $\neg_U \neg_U eq_I(i,j) = \{\top_D\} \cup \{d \in D \mid i = j \text{ and } d\top_D = \top_D\}$ is an equality predicate on I for $\mathscr{P}_U$, i.e.

$$\neg_U \neg_U eq_I(i,j) \vdash \varrho(i,j) \qquad \text{iff} \qquad \vdash \varrho(i,i)$$

Getting Boolean (2)

The boolean topos $\mathcal{E}_U$ is equivalent to the **classical realizability topos** $\mathcal{K}$ induced by the following **classical realizability structure**. Its set Λ of **terms** is D and its set Π of **stacks** is D^ω. **Application** is given by $(t_1 t_2)(\vec{s}) = t_1(t_2.\vec{s})$. The **control operator** is given by

$$\mathsf{cc}(t.\vec{s}) = t(\mathsf{k}_{\vec{s}}.\vec{s}) \quad \text{where} \quad \mathsf{k}_{\vec{s}}(t.\vec{r}) = t(\vec{s})$$

and one easily sees that $\mathsf{cc} \in P$. The **pole** is given by $t \perp\!\!\!\perp \vec{s}$ iff $t(\vec{s}) = \top$ and P is the set of **proof-like** terms.

Notice that $\neg_U A = (A.D^\omega)^{\perp\!\!\!\perp}$ and cc realizes $j_U A \to A^{\perp\!\!\!\perp\perp\!\!\!\perp}$. Thus, the tripos $\mathscr{P}_U$ is equivalent to the classical realizability tripos $\mathscr{P}_\mathcal{K}$ induced by the above classical realizability structure. It consists of families of biorthogonally closed subsets of D.

Thus $\mathcal{E}_U$ is equivalent to $\mathcal{K}$, the topos induced by $\mathscr{P}_\mathcal{K}$.

For D in Scott Domains we have $\mathcal{K} \simeq \mathbf{Set}$

Let D be the bifree solution of $D = \Sigma^{D^\omega}$ in Scott domains. Then $P \subseteq D$ contains a greatest element $\Phi = \bigsqcup P$ since $\vee : \Sigma \times \Sigma \to \Sigma$ ("parallel or") is Scott continuous.

Thus, for every biorthogonally closed $A \subseteq D$ we have that A is true, i.e. $A \cap P \neq \emptyset$, iff $\Phi \in A$. This allows us to show that

$$\models_{\mathcal{K}} A \to B \qquad \text{iff} \qquad \models_{\mathcal{K}} A \text{ implies } \models_{\mathcal{K}} B$$

and thus $\left(\mathscr{P}^I_{\mathcal{K}}, \vdash_I\right)$ is equivalent to $\left(\mathcal{P}(I), \subseteq\right)$.

Thus, we have $\mathcal{K} \simeq \mathbf{Set}$, i.e. nothing new.

Since "parallel or" is the culprit and it is not stable we now look at

$D \cong \Sigma^{D^\omega}$ in Coherence Spaces (1)

Since coherence spaces and stable linear maps are a model of classical linear logic we have $\Sigma^{D^\omega} = (!(D^\omega))^\perp$. This suggests the following concrete construction of D (inspired by Krivine).

Let S be the least set with $S = \mathcal{P}_{\mathsf{fin}}(\omega \times S)$. For $\alpha \in S$ and $n \in \omega$ let $\alpha_n = \{\beta \in S \mid \langle n, \beta \rangle \in \alpha\}$. By mutual recursion we define $|D|$ and $\frown$ as

$$\alpha \in |D| \quad \text{iff} \quad \forall n{\in}\omega. \forall \beta, \gamma \in \alpha_n. \, \beta \frown \gamma$$

$$\alpha \frown \beta \quad \text{iff} \quad \alpha \cup \beta \in |D| \text{ implies } \alpha = \beta.$$

The relation $\frown$ on $|D|$ is reflexive and symmetric. The domain D consists of cliques ordered by subset inclusion.

Notice that elements of D are *anti-chains* in $|D|$, i.e. $t \subseteq |D|$ such that for $\alpha, \beta \in t$, $\alpha \cup \beta \in |D|$ implies $\alpha = \beta$. One easily sees that elements of D^ω correspond to downward closed ideals in $(|D|, \subseteq)$.

$D \cong \Sigma^{D^\omega}$ in Coherence Spaces (2)

The ensuing pole is given by

$$t \perp\!\!\!\perp \vec{s} \qquad \text{iff} \qquad \exists \alpha \in t. \forall n \in \omega.\ \alpha_n \subseteq s_n$$

and application is given by

$$t_1 t_2 = \{\alpha \in |D| \mid \exists a \subseteq_{\mathsf{fin}} t_2.\ a.\alpha \in t_1\}$$

where $a.\alpha = (\{0\} \times a) \cup \{\langle n{+}1, \beta\rangle \mid \beta \in \alpha_n\}$.

With every $\alpha \in |D|$ we associate $|\alpha| \in \{0, 1\}$ recursively as follows

$$|\alpha| = 1 \quad \text{iff} \quad \exists n{\in}\omega. \exists \beta \in \alpha_n.\ |\beta| = 0$$

and define P as the set of all $t \in D$ with $|\alpha| = 1$ for all $\alpha \in t$.
It is easy to show that P meets its specification.

$\mathcal{K}$ not even Grothendieck

In the full paper we have shown that

1) subtoposes of $\mathcal{E}$ having small sums are equivalent to $\mathbf{Set}$

2) $\mathcal{K}$ is not equivalent to $\mathbf{Set}$

and, thus, the topos $\mathcal{K}$ cannot be a Grothendieck topos or even a forcing model.

$\mathbb{N}$ within D

Notice that $\top_D$ is an atom in D w.r.t. the stable order.
Thus, for $n \in \mathbb{N}$ there is a unique $\bar{n} \in D$ with

$$\bar{n}(\vec{s}) = \top \quad \text{iff} \quad s_n = \top_D$$

which is also atomic.

Notice that there is a least stable retract $r : D \to D$ below id_D fixing precisely $\bot_D$, $\top_D$ and $\bar{0}$.
Since i is a maximal element above $\bar{0}$ there is a least stable retract $\tilde{r} : D \to D$ fixing precisely $\bot_D$, $\top_D$ and i and sending $\bar{0}$ to i.

D universal for $\omega\mathbf{Coh}$

One knows from Asperti and Longo's book that $!(\mathbb{T}^\omega)$ is universal for $\omega\mathbf{Coh}$, i.e. countably based coherence spaces.

Since $\mathbb{T}$ embeds into D the coherence space $!(\mathbb{T}^\omega)$ embeds into $!(D^\omega)$. Thus $!(D^\omega)$ is universal for $\omega\mathbf{Coh}$.

Accordingly, we have that $D = (!(\mathbb{T}^\omega))^\perp$ is also universal for $\omega\mathbf{Coh}$.

$\mathcal{K}$ validates all true sentences of PA

The nno N of $\mathcal{K}$ has underlying set ω and

$$[\![n =_N m]\!] = {\uparrow}\top_D \cup {\uparrow}\{\bar{n} \mid n = m\}$$

Since for every $f : \omega \to D$ there is a $t_f \in D$ with $t_f\bar{n} = f(n)$ the topos $\mathcal{K}$ validates the ω-rule:
a sentence $\forall n{:}N.A(n)$ holds in $\mathcal{K}$ iff $A(n)$ holds in $\mathcal{K}$ for all $n \in \omega$.

Thus $\mathcal{K}$ validates all true arithmetic sentences in prenex normal form and thus all true sentences of first order arithmetic since $\mathcal{K}$ is boolean.

Finite Type Hierarchy over N in $\mathcal{K}$

Let 2 be the object with underlying set $2 = \{0, 1\}$

$$[\![i =_2 j]\!] = \uparrow\top_D \cup \uparrow\{\overline{i} \mid i = j\}$$

which is isomorphic to $2 = 1 + 1 \in \mathcal{K}$.

Since $\mathcal{K}$ is boolean we have $2 \cong \Omega_{\mathcal{K}}$.
Thus N contains $\Omega_{\mathcal{K}}$ as a subobject.
Accordingly, N^X contains $\mathcal{P}(X)$ as a subobject explaining why the finite type hierarchy over N is so difficult to grasp in classical realizability toposes.

Leibniz Equality and $\Delta : \mathbf{Set} \to \mathcal{K}$

In $\mathcal{K}$ for a set I equality on I is given by

$$eq_I(i,j) = \uparrow\top_D \cup \uparrow\{\bar{0} \mid i = j\}$$

since $\neg_U\neg_U\emptyset = \uparrow\top_D$ and $\neg_U\neg_U D = \{t \in D \mid t\top_D = \top_D\} = \uparrow\{\top_D, \bar{0}\}$.

Via $\tilde{r}$ the predicate eq_I is equivalent to the predicate

$$\widetilde{eq}_I(i,j) = \uparrow\top_D \cup \uparrow\{\mathsf{i} \mid i = j\} = \{\top_D\} \cup \{\mathsf{i} \mid i = j\}$$

since both $\top_D$ and i are maximal.

For every set I let $\Delta(I) = (I, \widetilde{eq}_I)$ giving rise to the well known **constant object** functor $\Delta : \mathbf{Set} \to \mathcal{K}$.

$\Delta(2)$ is Dedekind infinite

Krivine has shown (2012) that in $\mathcal{K}$ the object $\Delta(2)$ is infinite. But is it D-infinite, i.e. is there an injection $N \rightarrowtail \Delta(2)$?

If not this would imply that $\mathcal{K}$ does not validate Countable Choice which allows one to show that every infinite object is D-infinite.

But one can show – using bar recursion and induction - that $\mathcal{K}$ validates countable (and dependent) choice from which it follows that $\mathcal{K}$ validates the proposition that there is a 1-1 map from N to $\Delta(2)$.

However, there does not seem to exist a monomorphism $N \rightarrowtail \Delta(2)$ in $\mathcal{K}$ witnessing this existential statement.

Is $\mathcal{K}$ 2-valued?

Localic toposes are 2-valued iff they are equivalent to $\mathbf{Set}$. Since classical realizability toposes generalize forcing models they need not be 2-valued as is the case for realizability toposes.

For $D \cong \Sigma^{D^\omega}$ and P we don't know the answer. Notice that via *trace* elements of D correspond to antichains of **finite** elements in D^ω and elements of P to antichains whose intersection with PL^ω is empty.

If $A = A^{\perp\!\!\!\perp\perp\!\!\!\perp} \subseteq D$ contains an element of PL then $A^{\perp\!\!\!\perp} \cap P^\omega = \emptyset$.

Does the reverse implication hold?

A Partial Answer

We don't know in general but if $A^{\perp\!\!\!\perp} = \{t_n \mid n \in \omega\}^{\perp\!\!\!\perp}$ doesn't contain an element of P^ω then $A^{\perp\!\!\!\perp\perp\!\!\!\perp} \cap P \neq \emptyset$.

Proof
W.l.o.g. assume $t_{n+1}^{-1}(\top) \subseteq t_n^1(\top)$ then we get a tree all whose infinite paths land in the complement of P^ω. Since $D^\omega \setminus P^\omega$ is open we get a well-founded tree all whose leaves are in $D^\omega \setminus P^\omega$ which set is the trace of a $t \in \mathrm{PL}$ with $t^{-1}(\top) \supseteq A^{\perp\!\!\!\perp}$, i.e. $t \in A^{\perp\!\!\!\perp\perp\!\!\!\perp}$. $\square$

Some Open Questions

Presumably, full Axiom of Choice does not hold in $\mathcal{K}$. But we don't have a proof so far.

But, possibly, instances of Choice are not decided by $\mathcal{K}$

There is an obvious notion of computability for D since $|D|$ can be coded by $\mathbb{N}$ in such a way that $\frown$ is decidable. Computable elements are r.e. coherent subsets of $|D|$ w.r.t. this coding.

How much does this alter the ensuing classical realizability model?